$$E_t = \sum_i [Z_i^e A_i^e M_i^e e^{\Gamma_i^e z} + B_i^m M_i^m e^{\Gamma_i^m z}]$$
$$H_t = \sum_i [-A_i^e N_i^e e^{\Gamma_i^e z} + Y_i^m B_i^m N_i^m e^{\Gamma_i^m z}]$$
$$E_z = -(1/j\omega\hat{\epsilon}) \sum_i [A_i^e (k_i^e)^2 M_{zi} e^{\Gamma_i^e z}]$$
$$H_z = -(1/j\omega\mu) \sum_i [B_i^m (k_i^m)^2 N_{zi} e^{\Gamma_i^m z}] \tag{27}$$

where

$$\Gamma_i^e = \pm j \sqrt{[\hat{k}^2 - (k_i^e)^2] - \left(\frac{\mu\sigma v}{2}\right)^2} + \left(j\omega\Lambda + \frac{\mu\sigma v}{2}\right)$$
$$\Gamma_i^m = \pm j \sqrt{[\hat{k}^2 - (k_i^m)^2] - \left(\frac{\mu\sigma v}{2}\right)^2} + \left(j\omega\Lambda + \frac{\mu\sigma v}{2}\right) \tag{28}$$

and

$$Z_i^e = \frac{1}{Y_i^e} = \frac{1}{\omega\hat{\epsilon}}\left[\mp \sqrt{[\hat{k}^2 - (k_i^e)^2] - \left(\frac{\mu\sigma v}{2}\right)^2} - j\frac{\mu\sigma v}{2}\right]$$
$$Y_i^m = \frac{1}{Z_i^m} = \frac{1}{\omega\mu}\left[\mp \sqrt{[\hat{k}^2 - (k_i^m)^2] - \left(\frac{\mu\sigma v}{2}\right)^2} + j\frac{\mu\sigma v}{2}\right]. \tag{29}$$

Thus, the electromagnetic fields produced by an arbitrary distribution of sources in a uniform waveguide of arbitrary cross section filled with a moving medium, can be determined from (2), (5), and (14)–(19).

Let us next obtain the complementary solutions for the mode voltages and currents, i.e., the solutions of (10) and (11) for the homogeneous case:

$$\frac{dV_i^e}{dz} - (\alpha_i^e/j\omega\hat{\epsilon})I_i^e = 0$$
$$\frac{dI_i^e}{dz} - \mu\sigma v I_i^e + j\omega\hat{\epsilon}V_i^e = 0 \tag{21}$$
$$\frac{dV_i^m}{dz} + j\omega\mu I_i^m = 0$$
$$\frac{dI_i^m}{dz} - \mu\sigma v I_i^m - (\alpha_i^m/j\omega\mu)V_i^m = 0. \tag{22}$$

To determine explicit solutions of (21) and (22), it is convenient to eliminate either V_i or I_i yielding the one dimensional equations:

$$\frac{d^2V_i}{dz^2} - \mu\sigma v\frac{dV_i}{dz} + \alpha_i V_i = 0$$

or

$$\frac{d^2I_i}{dz^2} - \mu\sigma v\frac{dI_i}{dz} + \alpha_i I_i = 0 \tag{23}$$

where the superscript distinguishing the mode type has been omitted for simplicity, since the equations are of the same form for both modes. Thus, the complementary solutions are written in the following form:

$$I_i = A_i e^{\gamma_\pm z}, \quad V_i = B_i e^{\gamma_\pm z} \tag{24}$$

where A_i and B_i are constants and $\gamma_\pm$ are given in (16) or (19). The characteristic impedances and admittances can be defined by

$$Z_i = \frac{V_i}{I_i}, \quad Y_i = \frac{I_i}{V_i}. \tag{25}$$

For the homogeneous case, (12) is reduced to

$$V_{zi} = -(I_i^e/j\omega\hat{\epsilon})$$
$$I_{zi} = -(V_i^m/j\omega\mu). \tag{26}$$

From expressions (24)–(26), the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ within a source-free region of a waveguide filled with a moving medium can be expressed as follows:

The above expressions include, as their special cases, the results obtained by Collier and Tai,[1] who have derived, by the method of vector potentials, the electromagnetic fields within a source-free region of a circular or rectangular waveguide.

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Synthesis of Particular Unit Real Functions of Reflection Coefficient

Physically realizable input reflection coefficients of circuits consisting of commensurable lossless transmission lines and resistors have been described in terms of the parameter $z = e^{-j[(f/f_0)\pi]}$ where f is frequency and f_0 the frequency at which a line is a quarter-wavelength long, resulting in a function of z, $\rho(z)$, introduced in Young[1] and there called unit real (ur) functions. Later, the definition of ur functions was extended, and a test procedure was given for them.[2] In the original paper, it was shown that for a given unit real function $\rho(z)$, with reference characteristic impedance Z_0, and satisfying $\rho(0)\rho(\infty) = 1$, a unit element

$$Z_1 = Z_0\frac{1 + \rho(0)}{1 - \rho(0)}$$

could be removed leaving a reflection coefficient

$$\rho_1(z) = \frac{1}{2}\,\frac{\rho(z) - \rho(0)}{1 - \rho(z)\rho(0)} \tag{1}$$

which would be a lower degree than $\rho(z)$ in both numerator and denominator and which would still be ur.

For a given $\rho(z)$ which satisfies $\rho(0)\rho(\infty) = 1$, there is no guarantee that $\rho_1(0)\rho_1(\infty)$ will necessarily also be equal to unity, and so on, and therefore that the process of removal of unit elements will necessarily end in a resistive termination. The problem to be discussed in this correspondence is the form that $\rho(z)$ must have so that it will be capable of realization by cascaded transmission lines and a resistive termination. From this discussion, a procedure which tests any $\rho(z)$ to see whether or not it has the required form will be derived.

Obviously, from the work previously referred to, it is necessary that $\rho(z)$ be ur and therefore this must be the first test in the procedure. This condition by itself is not sufficient. This is demonstrated by the fact that the input $\rho(z)$ of a cascade of any combination of unit elements, stubs and resistors, a general form not allowed for the restricted problem under consideration, is still ur. For the general form, both the magnitude of $\rho(+1)$, the $\lambda/2$ line condition, or the magnitude of $\rho(-1)$, the $\lambda/4$ line condition, will be equal to unity, whereas for the desired form of a cascade of unit elements terminated in a pure resistor $\rho(\pm 1)$ is a real number less than unity. Hence, a second necessary condition is that $|\rho(\pm 1)| \neq 1$. Let

$$\rho(z) = \frac{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}{b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0} \tag{2}$$

from which, for $\rho(0)\rho(\infty) = 1$,

$$\frac{a_0 a_n}{b_0 b_n} = 1. \tag{3}$$

A special case of (3) occurs when

$$\frac{a_n}{b_n} = \frac{a_0}{b_0} = \pm 1 \tag{4}$$

and if coefficients of (2) are made to conform to this condition and the resulting $\rho(z)$ is substituted into (1), it will be seen that there results a value of $\rho_1(z)$ for which the degree of the denominator is one greater than that of the numerator and for which therefore $\rho_1(0)\rho_1(\infty) \neq 1$. This means that in this case the process of reduction of degree of input reflection coefficient by removal of cascade lines cannot be continued further. The condition that $\rho(0) \neq \rho(\infty)$ therefore constitutes a third necessary condition.

If the successive removal of unit elements is investigated by repeated substitution of (2) into (1), the conditions being imposed at any kth stage that $\rho_k(0)\rho_k(\infty) = 1$, but $\rho_k(0) \neq \rho_k(\infty)$, it is easily found that if the original $\rho(z)$ is of order n in both numerator and denominator, it is necessary that the following relationships between the coefficients of $\rho(z)$ must be satisfied simultaneously:

$$\left.\begin{aligned}
&1)\ a_0 a_n = b_0 b_n \\
&2)\ a_0 a_{n-1} + a_1 a_n = b_0 b_{n-1} = b_1 b_n \\
&3)\ a_0 a_{n-2} + a_1 a_{n-1} + a_2 a_n = b_0 b_{n-2} \\
&\qquad + b_1 b_{n-1} + b_2 b_n \\
&n)\ a_0 a_1 + a_1 a_2 + \cdots + a_{n-1} a_n = b_0 b_1 \\
&\qquad + b_1 b_2 + \cdots + b_{n-1} b_n.
\end{aligned}\right\} \tag{5}$$

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[1] L. Young, "Unit-real functions in transmission-line circuit theory," *IRE Trans. on Circuit Theory*, vol. CT-7, pp. 247–250, September 1960.

[2] C. S. Gledhill, "Resistor transmission-line circuits," *Proc. IEE (London)*, vol. 112, p. 2046, November 1965.

To summarize, therefore, if a given $\rho(z)$ is to be capable of physical realization as a cascade of lossless transmission lines terminated by a resistor the following conditions must be satisfied.

1) $\rho(z)$ must be a unit real.
2) $|\rho(\pm 1)| \neq 1$.
3) $\rho(0) \neq \rho(\infty)$.
4) The coefficients of the numerator and denominator polynomials of $\rho(z)$ must satisfy (5).

Conditions 1) and 4) above are both extensions of Young's necessary and sufficient conditions for the removal of a unit element with consequent reduction in the degree of $\rho(z)$ and they are in themselves sufficient. Conditions 2) and 3) are in the nature of tests which can be rapidly applied to eliminate forms of $\rho(z)$ impossible for the realization desired.

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Coupling to a Desired Wave

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Technical situations often arise where a wave of given characteristics must be propagated from a transmitting antenna to a receiving antenna. An example is shown in Fig. 1, where a Goubau wave must be launched along a dielectric-clad metallic conductor. The problem then arises of finding out how efficiently the wave is launched, i.e., how much of the total radiated power is propagated in the desired form. In this correspondence we attempt to give a very general definition of this coupling coefficient. Figure 2 illustrates a typical structure in which a radiating system is bounded by a surface S_1 consisting of a conducting wall S'' and a radiating aperture S'. In particular, S' can be closed surface completely surrounding the sources $\overline{J}$. The desired wave is characterized by a tangential electric field $\overline{E}_{t1}$ on S'. The actual tangential field is $\overline{E}_t$, and our problem consists of splitting $\overline{E}_t$ as

$$\overline{E}_t = \lambda \overline{E}_{t1} + \overline{E}_{t2}. \qquad (1)$$

The complex number λ represents the "launching coefficient" of the desired wave. In the language of functional analysis, $\lambda \overline{E}_{t1}$ is the projection of the actual wave (uniquely determined by its $\overline{E}_t$) on the subspace formed by the desired wave. To separate the two parts, the desired wave $\lambda \overline{E}_{t1}$ should be orthogonal, in some sense, to the complementary wave $\overline{E}_{t2}$. It is therefore necessary to introduce a suitable definition of the scalar product. This definition should lead to a splitting such that the sum of the powers carried by the individual terms in (1) is equal to the total radiated power. As "power orthogonality" is involved, the definition of the scalar product must necessarily contain the tangential com-

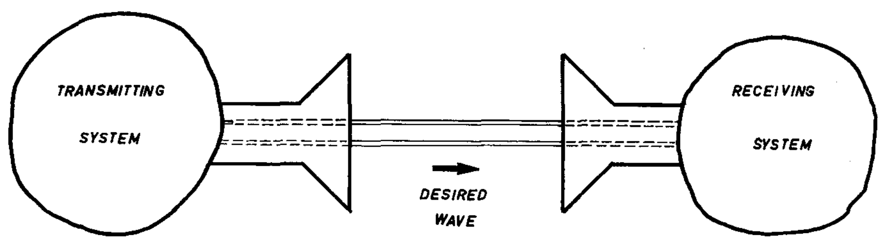

Fig. 1. Launching of a Goubau wave.

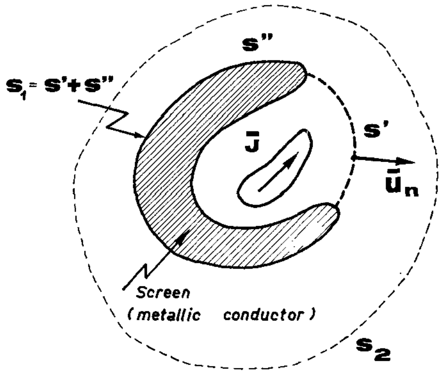

Fig. 2. A typical radiating system.

ponents of $\overline{E}$ and $\overline{H}$ on S', which can be collectively represented by a four-vector $\bar{\epsilon}$. A suitable definition for the product of waves whose four vectors are, respectively, $\bar{\epsilon}_a$ and $\bar{\epsilon}_b$ is

$$\langle \bar{\epsilon}_a, \bar{\epsilon}_b \rangle = \frac{1}{4} \iint_{S'} \bar{u}_n \cdot (\overline{E}_{ta}^* \times \overline{H}_{tb} + \overline{E}_{tb} \times \overline{H}_{ta}^*) dS. \qquad (2)$$

It is immediately apparent that the scalar product $\langle \bar{\epsilon}_a, \bar{\epsilon}_a \rangle$ of a wave with itself is the (real) average power radiated by the wave. It is also apparent that the scalar product of two waves depends, in general, on the boundary surface used to evaluate $\langle \bar{\epsilon}_a, \bar{\epsilon}_b \rangle$. Under certain circumstances, however, the value remains the same for two surfaces such as S_1 and S_2 in Fig. 2. This is true when:

1) the medium between S_1 and S_2 is Hermitian (condition: $\epsilon = \epsilon \dagger$ and $\mu = \mu \dagger$); example: a lossless plasma.
2) there are, in addition, no power sources between S_1 and S_2 (i.e., $\overline{J} = 0$ for both waves).

The property is easily proved by integrating

$$\begin{aligned} &\operatorname{div} (\overline{E}_a^* \times \overline{H}_b + \overline{E}_b \times \overline{H}_a^*) \\ &= \overline{H}_b \cdot \operatorname{curl} \overline{E}_a^* - \overline{E}_a^* \cdot \operatorname{curl} \overline{H}_b + \overline{H}_a^* \\ &\quad \cdot \operatorname{curl} \overline{E}_b - \overline{E}_b \cdot \operatorname{curl} \overline{H}_a^* \\ &= \overline{H}_b \cdot (j\omega\mu^* \cdot \overline{H}_a^*) - \overline{E}_a^* \cdot (j\omega\epsilon \cdot \overline{E}_b) \\ &\quad + \overline{H}_a^* \cdot (-j\omega\mu \cdot \overline{H}_b) - \overline{E}_b \cdot (-j\omega\epsilon^* \cdot \overline{E}_a^*) \\ &= 0 \end{aligned}$$

between S_1 and S_2. Assume, in particular, that S_2 is the sphere at infinity. The fields on that sphere have the general form

$$\overline{E} = \overline{F} \frac{e^{-jkR}}{R}$$

and

$$\overline{H} = \frac{1}{R_0} (\bar{u}_R \times \overline{E})$$

with

$$R_0 = (120\pi)\Omega.$$

For such case,

$$\langle \bar{\epsilon}_a, \bar{\epsilon}_b \rangle = \frac{1}{2R_0} \iint \overline{F}_a^* \cdot \overline{F}_b d\Omega. \qquad (3)$$

This equation shows that $\langle \bar{\epsilon}_a, \bar{\epsilon}_a \rangle$ is not only real, but also non-negative when the external medium satisfies the conditions enunciated above. It also shows that fields which have power orthogonality on a given surface have radiation vectors $\overline{F}_a$ and $\overline{F}_b$ satisfying

$$\iint \overline{F}_a^* \cdot \overline{F}_b d\Omega = 0. \qquad (4)$$

Having thus explored some of the properties of the power scalar product, we are now in a position to determine λ in (1). Use of the orthogonality requirement for $\bar{\epsilon}_1$ and $\bar{\epsilon}_2 = \bar{\epsilon} - \lambda \bar{\epsilon}_1$ yields

$$\lambda = \frac{\langle \bar{\epsilon}_1, \bar{\epsilon} \rangle}{\langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle} \cdot \qquad (5)$$

It is to be noticed that the "desired wave" is determined to within a constant factor. To lift the indeterminacy, one often adopts the "unit power normalization" $\langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle = 1$. It is now easy to show, with the help of the orthogonality property, that

$$\langle \bar{\epsilon}, \bar{\epsilon} \rangle = |\lambda|^2 \langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle + \langle \bar{\epsilon}_2, \bar{\epsilon}_2 \rangle. \qquad (6)$$

The total radiated power is, therefore, the sum of the powers in $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$, as required. The "power launching efficiency" is

$$\rho = \frac{|\lambda|^2 \langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle}{\langle \bar{\epsilon}, \bar{\epsilon} \rangle} = \frac{\langle \bar{\epsilon}_1, \bar{\epsilon} \rangle \langle \bar{\epsilon}, \bar{\epsilon}_1 \rangle}{\langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle \langle \bar{\epsilon}, \bar{\epsilon} \rangle} \leq 1. \qquad (7)$$

These various quantities are normally dependent on the choice of the "launching" aperture S', unless the conditions stated above for the constancy of the scalar product are satisfied.

At this point, we have solved the problem of determining the amplitude of the desired wave, $\bar{\epsilon}$ being given. However, knowledge of $\overline{E}_t$ alone, and not of $\bar{\epsilon}$, should be sufficient to determine the field uniquely. Furthermore, $\overline{E}_t$ is normally unknown, and must be determined by use of the boundary conditions across S' combined with a knowledge of the sources inside and outside S_1. The resulting "coupled regions" problem is a very difficult one to solve. This formulation can be clari-